S-Wave π-π Scattering Amplitudes in the New Form of the Strip Approximation*

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(Received 30 August 1965)

The new form of the strip approximation devised by Chew is applied to the calculation of the S-wave π-π scattering amplitudes, with forces produced by the exchange of ρ, P, and P' trajectories, and with Regge asymptotic behavior built in. Because of the sensitivity of the S wave to short-range forces not included in the strip approximation, a subtraction is made at the symmetry point, so that an extra free parameter λ is introduced, in addition to the strip width. Self-consistency is imposed on the S waves in the direct and crossed channels; λ is related to the scattering lengths for isospin 0 and 2 (a_0 and a_2), and it is concluded that solutions with $\lambda\approx-0.1$ giving $a_0\approx1$ and $a_2\approx0.2$ are in the best agreement with experiment. But some discussion is also given to the possibility that a_0 is negative, there being a bound-state pole of vanishing residue, which should correspond to the P' trajectory's crossing angular momentum zero.

I. INTRODUCTION

In their original paper on the π-π scattering amplitude Chew and Mandelstam[1] proposed a method of imposing self-consistency on the lower partial-wave amplitudes of the direct and crossed channels and absorbing the high-energy behavior into a single subtraction parameter λ, which was related to the values of the various isotopic-spin amplitudes at the symmetry point $s=t=u=\frac{4}{3}m_\pi^2$. Their method had two disadvantages: the partial-wave sum in the crossed channels does not converge in the direct channel outside the Lehmann ellipse, so that only singularities which are "near-by" in both the variables s and t could be included; and when it became known that there were resonances in the π-π system, in particular the ρ resonance, then the N/D equations required a cutoff. This introduced another parameter connected with the high-energy behavior, but whose relation to λ was not known.

In the intervening years much has been understood about both these problems, the first in terms of the strip approximation[2] which includes singularities which are nearby in one variable but not necessarily in both, and the second, in terms of continuation in angular momentum,[3,4] and the fact that Regge poles in the crossed channels control the asymptotic behavior in the direct channel.

Recently Chew and Jones[5] have put forward a new form of the strip approximation which incorporates both these ideas in a set of bootstrap equations for π-π scattering. The only free parameter of this scheme is the width of the strip, s_1, the boundary of which marks the point at which the resonance region is joined to the region governed by Regge asymptotic behavior.

Some preliminary results of bootstrapping trajectories have been reported,[6] and in principle, if one wishes to know about the S wave one has only to find a self-consistent set of trajectories and then project out the appropriate partial wave from the amplitude. However, the S wave, involving as it does an unweighted average over all momentum transfers, is very sensitive to the short-range forces, not all of which are included in the strip-approximation. In other words, one can expect the S wave to be much more sensitive to the inner regions of the double spectral functions than the higher partial waves.

In this paper we follow Chew and Mandelstam[1] in using a subtraction, and imposing self-consistency on the S waves in the direct and crossed channels, but we shall include the resonances as Regge poles, and impose Regge asymptotic behavior. Thus, apart from s_1, to which the solutions are insensitive, and the Regge trajectory functions which we suppose to be known,[7] our results depend only on one free parameter λ which will be related, dynamically, to the $I=0$, S-wave scattering length a_0. If this one piece of information is given to us the solution is determined. In practice, of course, there are considerable ambiguities in our input. The Regge parameters are not well known, but it turns out that the solution is not greatly affected by the choice made, providing that unitarity is satisfied. However, the choice of λ is crucial.

In the next section we discuss the experimental situation, and various theoretical suggestions as to how the S-wave amplitude should behave. In Sec. III we review the N/D equations, and in the following section, describe the calculation of the "potential" and our choice of Regge parameters. We conclude with a discussion of the results as a function of λ.

Solutions with λ around -0.1, giving an $I=0$ scattering length of about 1.0 and an $I=2$ scattering length of about 0.2, seem to be in accord with the present

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960). For a review of more recent work along similar lines, see K. Kang, *ibid.* **134**, B1324 (1964).

[2] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961).

[3] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1202 (1962).

[4] G. F. Chew, Phys. Rev. **129**, 2363 (1963).

[5] G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964).

[6] P. D. B. Collins and V. L. Teplitz, Phys. Rev. **140**, B663 (1965).

[7] R. J. N. Phillips and W. Rarita, Phys. Rev. **139**, B1336 (1965), and Phys. Rev. Letters **14**, 502 (1965). We assume that the P' passes through the newly discovered resonance at 1525 MeV reported by V. E. Barnes *et al.*, Phys. Rev. Letters **15**, 322 (1965).

experimental evidence, but solutions with negative scattering lengths, which have a "bound-state" ghost pole corresponding perhaps to the P' trajectory where it cuts angular momentum zero, are also considered. (The scattering lengths are expressed in pion Compton wavelengths throughout.)

II. PRESENT EVIDENCE CONCERNING THE S WAVE

As yet there is no clear evidence as to the nature of the $I=0$, S-wave scattering amplitude, but several indications tend to a similar conclusion, namely, that the scattering length is large and positive.

Abashian *et al.*[8] observed an anomalous peak in reactions such as

$$p+d\rightarrow \mathrm{He}^3+\pi^++\pi^-,$$

which they explained as being due to a large $I=0$ scattering length, and Booth and Abashian[9] find its value to be $a_0=2\pm1$. Hamilton *et al.*[10] have determined the contribution of the π-π interaction to the partial-wave dispersion relations for

$$\pi+N\rightarrow 2\pi+N,$$

and require $a_0=1.3\pm0.4$. A discussion of this evidence is given in Ref. 11. More recently Kacser *et al.*[12] have obtained $a_0=1.0\pm0.3$ in trying to fit the Ke_4 decay spectrum.

Contrary to this, however, Rothe[13] has considered the forward dispersion relations for π-π scattering itself, dividing the amplitude into a Regge asymptotic region whose behavior is controlled by the P and P' trajectories, and a resonance region fitted with Breit-Wigner formulas, the two regions being matched at $s=200m_\pi^2$. He finds $a_0=-1.7_{-0.5}^{+1.3}$. Also Kreps *et al.*[14] have found -1.72 in a dynamical calculation. It is hoped that a more definite value will be available soon using the method of Cabibbo and Maksymowicz.[15]

There is some evidence for at least two S-wave resonances. Brown and Singer[16] have proposed a 400-MeV particle, σ, of width 75–100 MeV, to explain the 3π decay modes of the η and K mesons, and this may have been observed by Samios *et al.*[17] The asymmetry of the neutral ρ in decay

$$\pi^-+p\rightarrow \pi^++\pi^-+n$$

has been discussed by Islam and Piñon[18] in terms of interference with a 60°, $I=0$, S-wave phase shift at 750 MeV, and by Durand and Chiu[19] in terms of an $I=0$, S-wave resonance with a mass and width similar to those of the ρ. Recently Feldman *et al.*[20] have reported evidence for such a particle at 700 MeV.

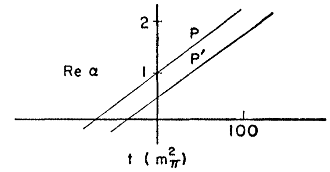

FIG. 1. The $I=0$ trajectories.

However, S-wave resonances are rather difficult to understand, and a calculation of the sort which we are proposing is certainly not able to produce them because of the way in which we treat short-range forces by means of a subtraction, and neglect higher threshold channels.

An alternative hypothesis has been put forward by Chew,[21] that the S-wave phase shift is in fact falling, and that the peaks observed or postulated are not resonances but occur when the phase shift is falling through an odd half-integer multiple of π. The reason for believing that this may be so is that the Pomeranchuk (P) and secondary Pomeranchuk (P') trajectories might well pass through angular momentum $\alpha=0$ (Fig. 1). Indeed the parameters for these trajectories found by Phillips and Rarita[7] do cut $\alpha=0$, and they say the trajectories might well have been represented by straight lines.

So if the P trajectory passes through $s=80m_\pi^2$ at $\alpha=2$ corresponding to the f^0, and through the Froissart limit of $\alpha=1$ at $s=0$, we may expect it to cut $\alpha=0$ near $s=-80m_\pi^2$, and similarly if the P' passes through the newly discovered particle at $120m_\pi^2$ with $\alpha=2$ and has an intercept[7] of $\alpha=0.5$ at $s=0$, then it may cut $\alpha=0$ at $s\approx-40m_\pi^2$, though if the trajectories have much curvature these points could be further to the left. Of course, at the point where the trajectory cuts $\alpha=0$ its residue must vanish, or there would be bound states in the physical region of the crossed channels, and at present we have no understanding of the mechanism which causes this vanishing.

Levinson's theorem[22] implies that for each trajectory that cuts $\alpha=0$ the phase-shift $\delta(s)$ at threshold is increased by a factor π over its value at ∞. Thus if we

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[17] N. P. Samios *et al.*, Phys. Rev. Letters **9**, 139 (1962).
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[20] M. Feldman *et al.*, Phys. Rev. Letters **14**, 869 (1965).
[21] G. F. Chew, Lawrence Radiation Laboratory (private communication).
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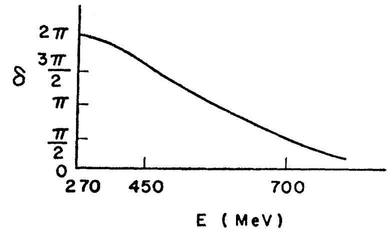

FIG. 2. Possible form for $I=0$, S-wave phase shift versus energy.

normalize $\delta(\infty)=0$, then we expect $\delta(4)=2\pi$ if both P and P' cut $\alpha=0$. The D function[23] would have two zeros below threshold at the points through which the trajectories pass, and two further zeros (of the real part of D) above threshold as the phase shift comes down through $\frac{3}{2}\pi$ and $\pi/2$. See Figs. 2 and 3. The fact that the N function would also have to vanish at the two zeros below threshold, so that real bound states do not occur, does not affect this argument. Such a situation would require a negative scattering length $[=N_0(4)/\mathrm{Re}D_0(4)]$. In fact, as we shall see, it has only proved possible to produce a D function with one zero below threshold.

III. THE SUBTRACTED N/D EQUATIONS

We set the S partial-wave amplitude[24]

$$A_0(s)=N_0(s)/D_0(s)\,, \tag{III.1}$$

where $N_0(s)$ has the left-hand cut of $A_0(s)$, and its right-hand cut for $s>s_1$; and $D_0(s)$ has the right-hand unitary cut for $4<s<s_1$, s_1 being the boundary of the strip. We can write the usual dispersion relations[4]

$$N_0(s)-B_0^v(s)D_0(s)=-\frac{1}{\pi}\int_4^{s_1}ds'\frac{\mathrm{Im}[B_0^v(s')D_0(s')]}{s'-s} \tag{III.2}$$

and

$$D_0(s)=1+\frac{1}{\pi}\int_4^{s_1}ds'\frac{\mathrm{Im}[D_0(s')]}{s'-s}\,, \tag{III.3}$$

where $B_0^v(s)$ is the "potential" function to be discussed in the next section. With elastic unitarity

$$\mathrm{Im}D_0(s^0)=-\rho_0(s)N_0(s)\,, \tag{III.4}$$

where $\rho_0(s)$ is the phase-space factor and is equal to

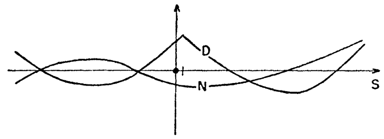

FIG. 3. Hypothetical N and D functions corresponding to Fig. 2.

$[(s-4)]/s]^{1/2}$. Setting

$$N_0(s_0)=A$$

and

$$D_0(s_0)=1\,, \tag{III.5}$$

and combining (III.2), (III.3), and (III.4), we obtain the subtracted equations

$$N_0(s)=A+B_0^v(s)-B_0^v(s_0)+\frac{(s-s_0)}{\pi}\times\int_4^{s_1}\frac{B_0^v(s')-B_0^v(s)}{(s'-s)(s'-s_0)}\rho_0(s')N_0(s') \tag{III.6}$$

and

$$D_0(s)=1-\frac{(s-s_0)}{\pi}\int_4^{s_1}\frac{ds'\rho_0(s')N_0(s')}{(s'-s)(s'-s_0)}. \tag{III.7}$$

Thus, given a potential function $B_0^v(s)$, we can solve the integral equation (III.6) for $N_0(s)$ and then use (III.7) for $D_0(s)$.

As in Ref. 1 we take s_0 to be the symmetry point $s_0=t_0=u_0=\frac{4}{3}$. The isotopic-spin amplitudes at the symmetry point are related by[1]

$$A^0=-5\lambda,\quad A^1=0,\quad A^2=-2\lambda\,, \tag{III.8}$$

the superscripts referring to isotopic spin, and λ being some constant. We shall make the approximation of setting the amplitude at the symmetry point equal to its lowest partial wave. In particular,

$$A^0(s_0,t_0,u_0)=A_0^0(s_0)\ \text{and}\ A^2(s_0,t_0,u_0)=A_0^2(s_0)\,. \tag{III.9}$$

Chew, Mandelstam, and Noyes[25] found that adding higher partial waves (principally the D wave in these cases) made very little difference.[26]

The subtraction constant $A^0\equiv N_0(s_0)/D_0(s_0)$ is dynamically related to the scattering length $a_0\equiv N_0(4)/\mathrm{Re}D_0(4)$, and could thus be fixed if we had a reliable estimate of the $I=0$ scattering length. Note, however, that because of the cusp in the D function at threshold, there may well be a considerable discrepancy between a_0 and A^0. In practice we shall express our results as functions of λ, and the relation (III.8) will be needed to relate the solutions for $I=0$ to those for $I=2$.

The integral equation (III.6) is solved by matrix inversion. Since the potential function $B_0^v(s)$ has a logarithmic singularity at $s=s_1$, the equation is not Fredholm. Chew[27] has shown how to transform the unsubtracted integral equation into a Fredholm form, and this transformation has been programmed.[28,29] However, this transformation cannot be used as it stands for the subtracted equation, and in any case it

[23] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963); R. J. Warnock, *ibid.* **131**, 1320 (1963).

[24] Our notation follows that of Refs. 1 and 5. We use units of $m_\pi=1$ throughout.

[25] G. F. Chew, S. Mandelstam, and H. P. Noyes, Phys. Rev. **119**, 478 (1960).

[26] G. F. Chew, Lawrence Radiation Laboratory (private communication).

[27] G. F. Chew, Phys. Rev. **130**, 1264 (1963).

[28] V. L. Teplitz, Phys. Rev. **137**, B136 (1965).

[29] D. C. Teplitz and V. L. Teplitz, Phys. Rev. **137**, B142 (1965).

requires a large amount of computer time. Jones and Tiktopoulos[30] have shown that if the norm of the kernel (in this case the coefficient of the logarithmic singularity) is less than one then matrix inversion can still be used despite the non-Fredholm nature of the equation. This point is discussed further in the next section.

IV. THE POTENTIAL

The potential which we use has two parts: contributions from the exchange of Regge trajectories, i.e., ρ, P, and P'; and the force from the cross-channel S wave, which we include in a self-consistent way. Simply to add these two parts would be to include some contributions twice in the S-wave amplitude, but this double counting can be avoided by means of the "normalization" procedure of Chew and Teplitz.[31]

The isotopic-spin crossing matrix for π-π scattering is[1]

$$\beta^{II'}=\begin{bmatrix}1/3 & 1 & 5/3\\ 1/3 & 1/2 & -5/6\\ 1/3 & -1/2 & 1/6\end{bmatrix}, \tag{IV.1}$$

so we have

$$B_0^{v0}(s)=\tfrac{1}{3}B_0^{PN}(s)+\tfrac{1}{3}B_0^{P'N}(s)+B_0^{\rho}(s)+\tfrac{1}{3}B_0^{s0}(s)+(5/3)B_0^{s2}(s) \tag{IV.2}$$

and

$$B_0^{v2}(s)=\tfrac{1}{3}B_0^{PN}(s)+\tfrac{1}{3}B_0^{P'N}(s)-\tfrac{1}{2}B_0^{\rho}(s)+\tfrac{1}{3}B_0^{s0}(s)+\tfrac{1}{6}B_0^{s2}(s), \tag{IV.3}$$

where $B_0^{sI}(s)$ is the contribution from the cross-channel S wave of isotopic spin I, and the remaining contributions are from the exchange of the trajectories. The superscript N means that the trajectory contribution has been normalized.

Chew and Jones[5] have given formulas for calculating the contributions of the trajectories, and we have, from Ref. 6 Eq. (10),

$$\begin{aligned}B_0^{v}(s)=\frac{1}{4q_s^2}\int_{-4q_s^2}^{0}\Bigg\{&\Gamma(t)\int_{-4q_t^2}^{s_1}\frac{du'}{u'-s}P_{\alpha(t)}\left(-1-\frac{u'}{2q_t^2}\right)+\Gamma(t)\xi\int_{-4q_t^2}^{s_1}\frac{du'}{u'-u}P_{\alpha(t)}\left(-1-\frac{u'}{2q_t^2}\right)\\ &+\xi\int_{s_1}^{\infty}\Gamma(t')P_{\alpha(t')}\left(-1-\frac{u'}{2q_{t'}^2}\right)\frac{du'}{u'-u}-\xi\int_{s_1}^{\infty}\Gamma(t')P_{\alpha(t')}\left(-1-\frac{u'}{2q_{t'}^2}\right)\frac{du'}{u'-t}\\ &+\frac{\pi\Gamma(t)}{\sin\pi\alpha(t)}\left[\xi P_{\alpha(t)}\left(-1-\frac{s}{2q_t^2}\right)+P_{\alpha(t)}\left(1+\frac{s}{2q_t^2}\right)\right]\Bigg\}.\end{aligned} \tag{IV.4}$$

This is simpler than Ref. 6 Eq. (10) because of the restriction to even signature (in the s channel) and angular momentum 0. $\xi(=\pm1)$ is the signature of the exchanged trajectory (in the t channel), and

$$\Gamma(t)=[2\alpha(t)+1](-q_t^2)^{\alpha(t)}\gamma(t), \tag{IV.5}$$

$\alpha(t)$ being the trajectory function and $\gamma(t)$ the reduced residue.

For the ρ trajectory, which being of negative signature has no S-wave component (in the t channel), we can use (IV.4) as it stands. For the P and P', however, it is necessary to "normalize." The effect of this procedure will be described in greater detail in a forthcoming paper,[32] but in brief what we do is subtract from the potential its contribution at $s=0$.

Thus

$$B_{l_s(s)}^{iN}=\frac{1}{4q_s^2}\int_{4q_s^2}^{0}dt\,P_{l_s}\left(1+\frac{t}{2q_s^2}\right)\times[V^i(t,s)-V^i(t,0)], \tag{IV.6}$$

where $V^i(t,s)$ is the expression in brackets { } in (IV.4) for trajectory i.

$V(t,0)$ can be represented by an expansion of the t channel discontinuity in a partial-wave series. Thus[31]

$$V(t,s)=\frac{1}{\pi}\int_4^{s_1}\frac{dt'}{t'-t}D_t(t',s) \tag{IV.7}$$

$$=\frac{1}{\pi}\int_4^{s_1}\frac{dt'}{t'-t}\sum_{l_t}(2l_t+1)\times\mathrm{Im}A_{l_t}(t')P_{l_t}\left(1+\frac{s}{2q_{t'}^2}\right) \tag{IV.8}$$

and

$$V(t,0)=\frac{1}{\pi}\int_4^{s_1}\frac{dt'}{t'-t}\sum_{l_t}(2l_t+1)\,\mathrm{Im}A_{l_t}(t'). \tag{IV.9}$$

If the S wave dominates this sum, as one may establish by comparing it with the D wave represented by the f_0 resonance with the experimental mass and width, then

$$V(t,0)\approx\frac{1}{\pi}\int_4^{s_1}\frac{dt'}{t'-t}\mathrm{Im}A_0(t'). \tag{IV.10}$$

Thus by using B_0^{iN} instead of B_0^{i}, when i refers to the P or P' trajectories, we have removed the contribution of these trajectories to the S-wave discontinuity, and are free to add to the potential the full contribution of this partial wave determined self-consistently, i.e., by

[30] C. E. Jones and G. Tiktopoulos, Princeton University unpublished report, 1965.
[31] G. F. Chew and V. L. Teplitz, Phys. Rev. 137, B139 (1965).
[32] P. D. B. Collins, following paper, Phys. Rev. 142, 1163 (1966).

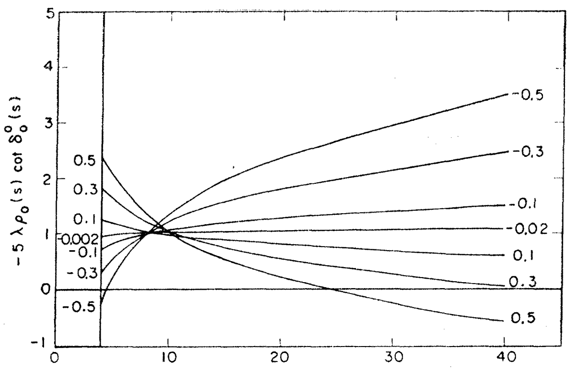

FIG. 4. $-5\lambda\rho_0(s)\cot\delta_0^0(s)$ versus s for the $I=0$, S-wave exchange alone.

setting

$$\mathrm{Im}A_{l_t}(s)=\mathrm{Im}A_{l_s}(s).$$

Then

$$B_0^{SI}(s)=\frac{2}{\pi}\int_4^{s_1}\frac{dt}{2q_s^2}Q_0\left(1+\frac{t}{2q_s^2}\right)\mathrm{Im}A_0^I(t)\quad\text{(IV.11)}$$

is the contribution to the potential from the isospin $I(=0$ or $2)$ S wave. The factor 2 comes from adding the contributions of the t and u channels.

We note that

$$\mathrm{Im}A_0(t)=\frac{\rho_0(t)}{\rho_0^2(t)+[\mathrm{Re}D_0(t)/N_0(t)]^2},\quad\text{(IV.12)}$$

and $D_0(t)$ and $N_0(t)$ are obtained as solutions to the N/D equations.

We have to choose forms for the Regge functions $\alpha_i(t)$ and $\gamma_i(t)$. The functions are real analytic in t cut from threshold to ∞,[4] but we only need to know them for $t<0$. We take a pole approximation, which has been found adequate in the past for α,[6] and should also serve for γ.[4] It turns out that the solutions are not very dependent on the precise choice of parameters. The results quoted in the following section use these functions:

$$\begin{aligned}
\alpha_\rho&=-1.5+2.0/(1-t/140),\\
\gamma_\rho&=0.01/(1-t/100)(24)^{1-\alpha(t)},\\
\alpha_P&=-1.0+2.0/(1-t/240),\\
\gamma_P&=0.007/(1-t/100)(24)^{1-\alpha(t)}\alpha(t),\\
\alpha_{P'}&=-1.752+2.251/(1-t/100),\\
\gamma_{P'}&=0.142/(1-t/100)(24)^{1/2-\alpha(t)}2\alpha(t).
\end{aligned}$$

These have been chosen so that

$$\begin{array}{llll}
\alpha_\rho=1 & \text{for } t=28; & \alpha_\rho=0.5 & \text{for } t=0,\\
\alpha_P=1 & \text{for } t=0; & \alpha_P=2 & \text{for } t=80,\\
\alpha_{P'}=0.5 & \text{for } t=0; & \alpha_{P'}=2 & \text{for } t=120,
\end{array}$$

corresponding to the known particle masses and the intercepts of Ref. 7. The remaining parameter, the pole position, was chosen to make the trajectories fall rapidly.

γ_ρ was chosen so that $\gamma_\rho(28)/\alpha_\rho'(28)$ corresponds[6] to a ρ width of 100 MeV.

$\gamma_P(0)$ is chosen so that the total cross section $\sigma_{\pi\pi}(\infty)=11$ mb and $\gamma_{P'}(0)$ is roughly the value implied by the findings of Ref. 7 and the factorization theorem. Again the pole positions were chosen so that the functions fell sufficiently rapidly with negative t. Multiplication by $\alpha(t)$ ensures that residues of P and P' vanish if $\alpha(t)=0$. By sufficiently rapidly we mean such as to give a logarithmic derivative of the potential corresponding to a π-π diffraction peak of inverse width ≈ 4 GeV^{-2}.[33] Apart from the desirability of satisfying this criterion so that the functions will correspond to the true physical values as closely as possible, it turns out also to be necessary if unitarity is not to be violated by the potential, making solution of the N/D equations impossible.

As has been discussed previously,[4,27] the potential calculated from (IV.4) has a logarithmic singularity at s_1, such that

$$B_0^v(s_1)\underset{s\to s_1}{\longrightarrow}-(1/\pi)\,\mathrm{Im}B_0^v(s_1)\ln(s_1-s)\quad\text{(IV.13)}$$

and

$$\mathrm{Im}B_0^v(s_1)=\sin^2\delta_0(s_1)/\rho_0(s_1),\quad\text{(IV.14)}$$

where $\delta_0(s)$ is the S-wave phase shift. Thus if unitarity is not to be violated at s_1 we require that

$$\rho_0(s_1)\,\mathrm{Im}B_0^v(s_1)\leqslant 1,\quad\text{(IV.15)}$$

or, since $s_1\gg 4$,

$$\mathrm{Im}B_0^v(s_1)\leqslant 1.\quad\text{(IV.16)}$$

Now[6]

$$\mathrm{Im}B_0^{vI}(s_1)$$

$$=\sum_i\frac{\beta^{II_i}}{4q_{s_1}^2}\int_{-4q_{s_1}^2}^0 dt\,\pi\Gamma_i(t)P_{\alpha_i(t)}\left(-1-\frac{s_1}{2q_t^2}\right),\quad\text{(IV.17)}$$

the sum running over the P, P', and ρ trajectories, and the expression is smaller, the larger are $\alpha'(t)$ and $\gamma'(t)$. With the parameters quoted from (IV.2) and (IV.3)

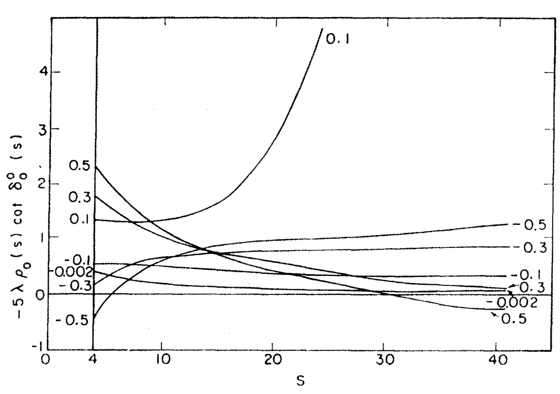

FIG. 5. $-5\lambda\rho_0(s)\cot\delta_0^0(s)$ versus s for exchange of elementary ρ of width $0.7m$ and $I=0$ S wave; $s_1=200$.

[33] G. F. Chew and V. L. Teplitz, Phys. Rev. 136, B1154 (1964).

we find, for $s_1=200$,

$$\mathrm{Im}B_0^{v0}(s_1)=0.9531$$

and

$$\mathrm{Im}B_0^{v2}(s_1)=0.4002\,,$$

which is close to the limit. If unitarity is violated at s_1 the norm of the kernel is greater than 1 and the N/D equations cannot be solved by matrix inversion.[30] Thus even though the behavior of the amplitude at s near s_1 is of no interest to us in these subtracted N/D equations, whose solutions do not depend on distant singularities, it is still necessary to ensure that the potential does not violate unitarity at s_1.

The contribution of the ρ to the potential is much greater than that of the P and P', and very similar results are obtained if the latter are neglected. It is also interesting to compare the effect of exchanging a ρ Regge trajectory with that of exchanging an elementary (fixed spin) ρ. It is well known[34,35] that there are ambiguous zero-range components in the fixed-spin potential which affect only the S wave. However, the approximation is often made of neglecting these zero-range parts and taking

$$B_0^{\rho\mathrm{el}}=3m_\rho g(1+s/2q_\rho^2)Q_0(1+m_\rho^2/2q_s^2)/q_s^2\,, \quad \text{(IV.18)}$$

where g is the width of the ρ (in m_π). This potential is rather different in its energy dependence from the Regge form, whose principal term is an average in t over $\Gamma(t)P_{\alpha(t)}[-1-s/2q_t^2]/\sin\pi\alpha(t)$ from $t=0$ to $-4q_s^2$, and which goes almost to zero at the symmetry point, whereas (IV.18) has these factors evaluated at $t=m_\rho^2$. This difference is not very important in our calculation because the low-s part of $B_0^v(s)$ comes mainly from $B_0^s(s)$. Also the subtraction will remove any dependence of the solution on the short-range components included in the Regge form.

It is possible to use (IV.11) as it stands to obtain the potential at the subtraction point, $B_0^s(s_0)$, but (IV.4) cannot be used for $s<4$. Instead we obtained $B_0^i(s_0)$ by extrapolation from the values for $s>4$.

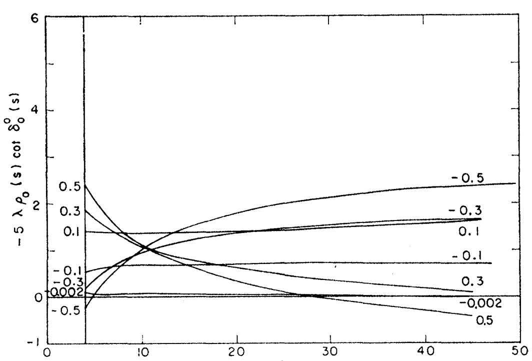

FIG. 6. $-5\lambda\rho_0(s)\cot\delta_0^0(s)$ versus s for exchange of Reggeized ρ, P, and P' with $I=0$ S wave; $s_1=200$.

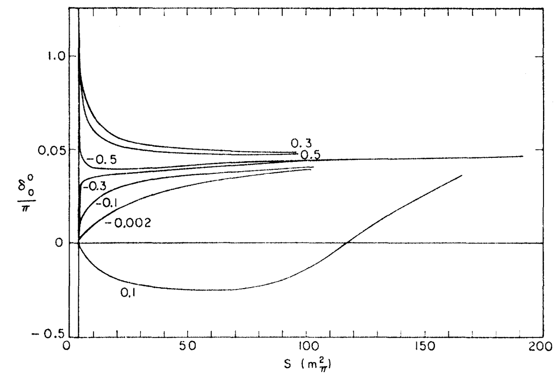

FIG. 7. The $I=0$, S-wave phase shift $\delta_0^0(s)$ versus s for the Reggeized ρ, P, P' input; $s_1=200$.

V. DISCUSSION OF THE RESULTS

Our method of solution is to begin with $B_0^{v0}(s)$ as given by the trajectory contributions to (IV.2) only. The N/D equations are then solved with a chosen value of λ, and $B_0^{s0}(s)$ calculated from (IV.11) and (IV.12). Adding this to $B_0^{v0}(s)$ a new solution was obtained, and after four or five such cycles the solution was stable. $B_0^{s2}(s)$ was neglected as small. The resulting form of $B_0^{s0}(s)$ was used in (IV.3) to obtain solutions for $I=2$.

For comparison we repeated the calculation with the S-wave potential alone, and also with the elementary ρ potential instead of the Regge potentials.

In Figs. 4–6 we plot the function $-5\lambda\rho_0(s)\cot\delta_0^0(s) = -5\lambda[\mathrm{Re}D(s)/N(s)]$ against s for a range of values of λ for the three different types of input potential. For the S wave alone, results very similar to those of Ref. 1 were obtained except for $\lambda=-0.5$ and -0.3, where the P wave gave an important contribution. The Regge input gives curves similar in form to those for elementary ρ except for $\lambda=0.1$, where the zero of the N function (and hence of the phase shift) occurs at a much lower energy for the latter input.

The phase shifts for the Regge case are plotted in Fig. 7. They are constrained in each case to $\delta(s_1)=0.43\pi$ by the imposition of Regge asymptotic behavior, and in fact closely approach this value for $s=100$ except for $\lambda=0.1$. This value of λ is close to that for which a bound state appears at $s=-\infty$ and $\delta_0^0(0)$ jumps from 0 to π. It is difficult to estimate closely the value of λ for which the bound state appears because we cannot calculate the D function with the requisite accuracy for very large negative s, but it certainly exists when λ has reached 0.14. This pole becomes less bound as λ increases, and reaches a position where it might be identified with the P' (say) for $\lambda=0.24$ to 0.30, i.e., $s=-109$ to -44.

[34] G. F. Chew, Lawrence Radiation Laboratory Report UCRL-16101, 1965 (unpublished).
[35] E. S. Abers and V. L. Teplitz, (unpublished CERN report, 1965).

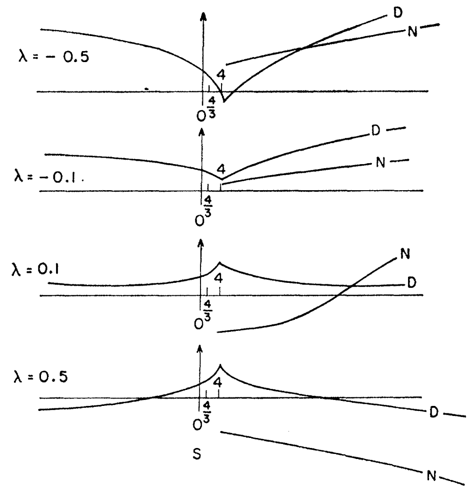

FIG. 8. Sketches of the N and D functions for the same case as Fig. 7. Not to scale.

Table I shows the bound-state positions and scattering lengths for given values of λ. To give an indication of the ambiguity resulting from our lack of knowledge of the input parameters we also give results for an elementary ρ with a cutoff at $s_1=400$, and with a width of $1m_\pi$ rather than the experimental value of $0.7m_\pi$. Changes of the Regge parameters of a similar magnitude produce similar variations.

For $\lambda=-0.5$ a bound state is produced in all cases, a result already noted in Ref. 1. As λ is increased, the bound state moves up to threshold and disappears, the scattering length then becoming positive. However, as λ is increased farther, the N function at threshold passes through zero, turning negative, and the scattering length becomes negative again. Finally for large λ ($\lambda > \approx 0.14$) a zero of D_0 develops at $s=-\infty$ and moves to the right. Since $D_0(s_0)$ is constrained to 1 it is clear that the zero cannot move to the right of s_0 as $\lambda \rightarrow \infty$. Figure 8 shows sketches of the N and D functions for various λ. In Figs. 9 and 10 we give corresponding solutions for $I=2$.

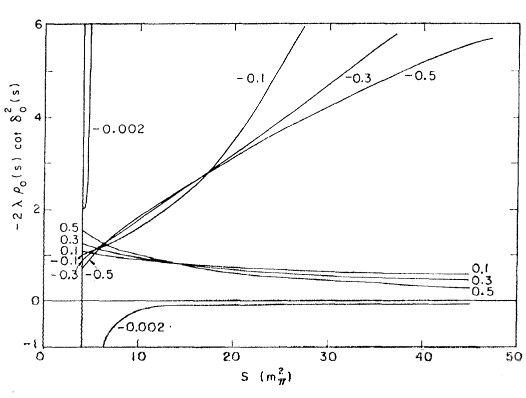

FIG. 9. $-2\lambda\rho_0(s)\cot\delta_0^2(s)$ versus s for exchange of Reggeized ρ, P, and P' the $I=0$ S wave and $I=2$ S wave; $S_1=200$.

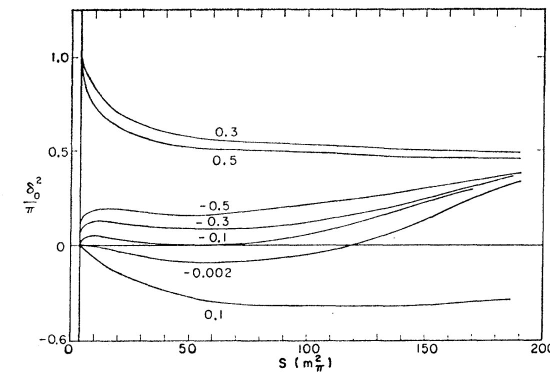

FIG. 10. The phase shifts corresponding to Fig. 9.

If it is true experimentally that the $I=0$, S-wave scattering length is approximately 1, then the correct solution should be that for $\lambda=-0.1$. The corresponding $I=2$ scattering length of 0.2 is in moderate agreement with Ref. 36, who estimate it to be 0.09.

If alternatively one believes that the scattering length is negative, the best solution seems to be for $\lambda \approx 0.3$, where the bound state at $s\approx -50$ might well be identified with the P' trajectory. The residue of this pole is large in that for the case of Fig. 7, we have, e.g., for

$$\lambda=0.3,$$

$$N_0(-44)=-1.36,$$

and

$$\left[\frac{d}{ds}D_0(s)\right]_{s=-44}=0.0075,$$

and the pole may be written

$$-181/(s+44).$$

TABLE I. Bound-state positions and scattering lengths.[a]

λ	−0.5	−0.3	−0.1	−0.002	0.1	0.3	0.5
Case			Bound-state positions				
a	3.96	⋯	⋯	⋯	⋯	−44.3	−13.2
b	3.85	⋯	⋯	⋯	⋯	−33.1	−8.76
c	8.85	⋯	⋯	⋯	⋯	−33.1	−8.76
d	3.68	⋯	⋯	⋯	⋯	−26.4	−6.5
e	3.86	⋯	⋯	⋯	⋯	−44.0	−11.7
f	3.86	⋯	⋯	⋯	⋯	−42.1	−12.2
			Scattering lengths				
a	−12.1	5.3	0.66	0.10	−0.396	−0.82	−1.05
b	−5.7	12.6	0.95	0.23	−0.37	−0.82	−1.04
e	−8.1	3.3	0.87	0.11	−0.35	−0.82	−1.05
g	1.41	0.67	0.19	0.002	−0.18	−0.46	−0.65

[a] The units are: bound-state positions $s(m_\pi{}^2)$, and scattering lengths (pion Compton wavelengths). The isotopic spins and forces for the various cases are:

a. $I=0$, S wave π-π alone;
b. $I=0$, S wave π-π with elementary ρ of width $0.7m_\pi$, $s_1=200$;
c. $I=0$, S wave π-π with elementary ρ of width $0.7m_\pi$, $s_1=400$;
d. $I=0$, S wave π-π with elementary ρ of width $1.0m_\pi$, $s_1=200$;
e. $I=0$, S wave π-π with Reggeized ρ, P and P', $s_1=200$;
f. $I=0$, S wave π-π with Reggeized ρ, P and P', $s_1=400$;
g. $I=2$, S wave π-π with Reggeized ρ, P and P', $s_1=200$.

[36] Aachen-Birmingham-Bonn-Hamburg-London (I.C.)–München Collaboration, Nuovo Cimento 31, 485 (1964).

But if the potential function were to have a suitable behavior to make $N(-44)=0$, the solution of the N/D equations in the strip region, $4<s<s_1$, would be very little changed because of the subtraction. Or, inverting this argument, our solution does not determine N for points a long way outside the strip with any accuracy. However, for this value of λ the $I=2$ amplitude also has a bound-state pole, as Fig. 10 shows, though again it is not possible to determine its position except that it is at $s<-300$. Since no $I=2$ trajectories are known, it seems that such solutions must be wrong. If the scattering length is to be -1.7 we require a very large $\lambda(\approx 40)$, and the $I=0$ bound-state position at $s=0.56$ is too close to the symmetry point to be identified with any known trajectory.

According to our present information the best solution is that with $\lambda=-0.1$.

ACKNOWLEDGMENTS

I would like to thank Professor G. F. Chew for many discussions and suggestions throughout the course of this work. I am also indebted to Dr. V. L. Teplitz for the use of his computer programs. The hospitality of the Lawrence Radiation Laboratory is gratefully acknowledged. The work was performed while I was the holder of a D.S.I.R. Research Fellowship.

PHYSICAL REVIEW VOLUME 142, NUMBER 4 FEBRUARY 1966

Inadequacies of the New Form of the Strip Approximation for the π-π Scattering Amplitude*

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(Received 10 September 1965)

The new form of the strip approximation is used to obtain mutually self-consistent trajectories with isopin $I=0$ and $I=1$ in the π-π system. However, these trajectories do not correspond to those which are obtained from experiment, and violate unitarity in the asymptotic region. The trajectories obtained from experiment, which satisfy unitarity, are shown not to produce sufficient strength to bootstrap themselves. Also the $I=0$ trajectory gives rise to a repulsive potential, and to obtain a solution of the N/D equations we are impelled to the doubtful assumption that this repulsion is completely cancelled by other $I=0$ trajectories that do not reach the right-half angular-momentum plane. It is concluded that both these difficulties stem from the fact that the potential is included only in the first Born approximation, and that more satisfactory results would be forthcoming if the potential were iterated in the way proposed by Mandelstam.

I. INTRODUCTION

THE new form of the strip approximation has been proposed[1,2] as a method of calculating scattering amplitudes in accordance with the principles of maximal analyticity of the first and second kinds. The amplitudes are constructed so that they satisfy the Mandelstam representation, and all their poles are Regge poles. Such amplitudes will have the correct behavior in the low-energy resonance region where the poles dominate, and also in the high-energy region where Regge asymptotic behavior is observed. It is hoped that these features include enough of the dynamics for the amplitudes to be self-consistent in the sense that the "potential" due to the crossed-channel singularities generates the direct-channel singularities.

For the π-π amplitude, in which identical processes occur in the direct and crossed channels, this self-consistency amounts to a "bootstrap" requirement. The dominant Regge trajectories, ρ, P, and P' should bootstrap themselves.

Chew and Jones[2] have devised a set of equations for investigating this possibility using the N/D method, with the N function having the cuts of the potential, and the D function the unitarity cut in the strip region. Results have already been reported[3] for a self-consistent ρ trajectory, but the ρ potential also generated an $I=0$ trajectory which was not included in the potential. In this paper we complete the solution by obtaining a pair of mutually self-consistent trajectories, one having $I=0$ and the other $I=1$. However, these trajectories have several unsatisfactory features, and we are led to discuss some deficiencies of the new form of the strip approximation, and how they might be rectified.

In the next two sections the N/D equations and the method of calculating the potential from the exchange of Regge trajectories are reviewed. The fourth section is devoted to a discussion of the potential for P exchange, which is repulsive. The total potential for $I=0$ exchange may be made attractive by means of a "normalization"

* This work was done under the auspices of the U. S. Atomic Energy Commission.

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[1] G. F. Chew, Phys. Rev. **129**, 2363 (1963).

[2] G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964).

[3] P. D. B. Collins and V. L. Teplitz, Phys. Rev. **140**, B663 (1965).